

A Family of Totally Rank One Two-sided Shift Maps

YUE WU,¹ DONGMEI LI,² YUNJIAN WANG,³ DIQUAN LI,⁴

Abstract

'Generalized del Junco-Rudolph's map', a sub-family of generalized Chacon's map ([3]), is introduced. A skew product related to the structure of the Generalized del Junco-Rudolph's map is introduced. A Relative Prime Relation, $h_{k+1} \equiv 1 \pmod q$ is verified, based on the proposition of iterations of this skew product. We say a measure-preserving transformation T is totally rank one if $T^n, n \in \mathbb{N}$ is rank one. In this paper, we show that of every Generalized del Junco-Rudolph's map is totally rank one.

Since Boltzmann introduced the term of 'Ergodic' in his work on statistical mechanics, the ergodic theory has been studied and developed actively regarding modeled physical systems with dynamic nature. From ergodicity, mixing properties, isomorphism of systems, to simplicity, and many other properties, we seriously desire concrete examples to better understand and study these properties with related classifications. The cutting-and-stacking method provides effective measure theoretical approximations of transformations, and at the same time, it provides a fruitful generating machine for typical examples and counter examples corresponding to symbolic shifting maps. Extending from the original construction by R.V. Chacon ([2]), A.A. Prikhod'ko and V.V. Ryzhikov (Construction 7 in [9]) further described how to construct a transformation on $[0, 1]$ using the cutting-and-stacking method. This kind of constructional approach may help us understand not only the symbolic representing of a dynamic system but also its powers. G.R. Goodson ([5]) studied the condition for an ergodic automorphism T to be conjugate to its composition square T^2 . What's more del Junco ([7]) showed that the property of being conjugate to its square, is a non-generic property of automorphisms. Thus therefore, it is not trivial at all to study the powers of ergodic automorphisms.

We say a finite measure-preserving system $(\mathbb{X}, \beta, T, \mu)$ is rank one if a series of Rokhlin tower with base subset F approximates any partition P of \mathbb{X} . Rank one system can be defined in multiple literatures ([3]). Rank one implies simple spectrum ([1],[4]), rigidity implies singular spectrum ([4]). The cutting and stacking structure also gives us a way to study a rank one system by the symbolic methods([2], [6],etc.) All these make it interesting to learn whether a transformation is rank one. Veech [10] showed that, measure theoretically, almost all interval exchange transformations are rank one. A famous rank one transformation, the Chacon's map coded as $B_{k+1} = B_k B_k 1 B_k$, and its extension are also studied. In this chapter we will study the family of generalized del Junco-Rudolph's maps, and extend the rank one property of this family of maps ([6]), to their powers of natural numbers. We admit the notion of totally rank one for this property (Definition 3.1), and make the main conclusion as Theorem 3.3.

¹Schlumberger WesternGeco Geosolution, Houston, Texas, USA

²Corresponding Author; Harbin University of Science and Technology, School of Applied Science, Harbin, Heilongjiang, China

³Schlumberger SWT Technology Center, Houston, Texas, USA

⁴Central South University, School of Geosciences and Info-physics, University, Changsha, Hunan, China

1 Generalized Del Junco-Rudolph's maps

In [6], an example described by symbolic recursion is given. We extend this map, notated by Del Junco-Rudolph's map here, to a family of symbolic maps-generalized Del Junco-Rudolph's maps. It is a sub-family of the family of generalized Chacon's map [3] We will describe the construction and some observations of the generalized Del Junco-Rudolph's maps in this section.

The recursion formula determines the language of the system, thus also the phase space of the system, which is a subset of \mathbb{Z} product space of the set of alphabet $0, 1$. That is:

$$\begin{aligned} B_0 &= 0 \\ B_1 &= (B_0)^a 1 (B_0)^{b-a}, (1 \leq a \leq b-1, b \geq 2) \\ &\dots \\ B_{k+1} &= (B_k)^{a \cdot b^k} 1 (B_k)^{(b-a) \cdot b^k} \end{aligned}$$

Let $X \subset \{0, 1\}^{\mathbb{Z}}$ be defined by $X = \{x = \dots x_{-1} x_0 x_1 \dots x_k \dots \mid \text{for any } m \in \mathbb{Z}, l \in \mathbb{N}\}$, $x_m x_{m+1} \dots x_{m+l-1}$ is a consecutive sequence of some B_k , $k \in \mathbb{N}$. Let T be the left-shifting transformation on X , so T is a two-sided shift map.

Definition 1.1. *If a sequence \bar{a} appears consecutively in some B_k , we say \bar{a} is a valid sequence or a word in \mathbb{X} (language) of length l .*

Next some combinatorial fact about T will be shown, some further concerns will be discussed in Chapter 4 for the special purpose. The following facts are easily seen or computed. Some of them have been pointed out in [del Junco, Rudolph 1].

Lemma 1.2. *Let T be the symbolic transformation defined above, i.e. the generalized Del Junco-Rudolph map, we have the follow propositions:*

1. $h_{k+1} = b^{k+1} h_k + 1$, and $b^{\frac{k(k+1)}{2}} < h_k < b^{\frac{k(k+1)}{2} + 1}$.
2. *except for the two given copies of B_n , there is no other consecutive sequence of letters which is identically B_n .*
3. *suppose $x|_m^{m+2h_k} = B_k 1 B_k$, then $x|_{m-h_k}^{m-1} = B_k = x|_{m+2h_k+1}^{m+3h_k}$, for any $k > 1$.*

Proof. (iii) Since $B_k 1 B_k$ is in some B_j , $j > k$, both the left and the right word to $B_k 1 B_k$ of length h_k are B_k . □

Topologically, we have the following conclusion:

Lemma 1.3. *The system (\mathbb{X}, T) is minimal.*

Proof. This is true based on the coding construction of the system. □

Measure theoretically a T -invariant measure of \mathbb{X} may be determined by the measure of each cylinder, which is the asymptotic density of the cylinder name in the language of the symbolic structure. Del Junco Rudolph's approximation would be introduced for this purpose, which assures the coincidence of the measure defined and the asymptotic density by the uniform ergodic theorem.

Remark. *We say A is a cylinder set with name $\bar{\beta}$ ($|\bar{\beta}| = l$) at position j , if $A = \{x \mid x_j \dots x_{j+l-1} = \bar{\beta}\}$.*

Let $d_k(\bar{a})$ be the density in $B_k^{\mathbb{Z}} = \cdots B_k B_k \cdots$ of the occurrences of a word $\bar{\beta} = a_0 \cdots a_{l-1}$. Then for k large enough:

$$\left| d_{k+1}(\bar{\beta}) - d_k(\bar{\beta}) \right| < 2l/h_{k+1} \quad (1.1)$$

This inequality is true because the only difference is taking place at the higher spacer (the spacer between two B_k).

Definition 1.4. Let $\mathcal{S}_k = \{x : x|_{-h_k}^{h_k} = B_k 1 B_k\}$.

Next the measure μ is introduced based on the notion of $d_k(\bar{\beta})$:

$$d(\bar{\beta}) = \lim_{k \rightarrow \infty} d_k(\bar{\beta}) \quad (1.2)$$

If A is a cylinder set with name $\bar{\beta}$, we define

$$\mu(A) = d(\bar{\beta}) \quad (1.3)$$

and extend this countably additive measure to a shift invariant Borel measure μ on $(\mathbb{X}, \mathcal{B}, T)$, where \mathcal{B} is the σ -algebra generated by the set of all the cylinder sets.

Lemma 1.5. μ is the unique invariant measure of $(\mathbb{X}, \mathcal{B}, T)$ up to a multiple.

Proof. Each point $x \in \mathbb{X}$ is a generic point of $(\mathbb{X}, \mu, \mathcal{B}, T)$. □

The measure of \mathcal{S}_k is bounded by:

Proposition 1.6. $\frac{1}{h_{k+1}} < \mu(\mathcal{S}_k) < \frac{b}{h_{k+1}}$

Proof. Occurrences of \mathcal{S}_k on the orbit of any $x \in \mathbb{X}$ are separated by at most h_{k+1} and at least $\frac{h_{k+1}+1}{b}$. □

Remark. The above statement is labeled as a proposition, since it is crucial for the future evaluation.

The rigidity of $(\mathbb{X}, \mathcal{B}, \mu, T)$ is easily seen by looking at $T^{h_k}(C)\Delta C$ for any cylinder C , typically just consider $B_{\Delta}^{n,k} = T^{h_n}(B_k)\Delta B_k, n \geq k$. The notion of coding distance provides us another way to understand the rigidity of $(\mathbb{X}, \mathcal{B}, \mu, T)$ and to evaluate the approximate speed of the power of T tending to identity along the sequence $\{h_k\}$.

Let $\xi^{N,k} = B_k^N$ and set

$$\delta_k^{(N)}(T^t) = \frac{1}{Nh_k} \sum_{i=1}^{Nh_k} d_{code}^{(N,k)}(i, t) \quad (1.4)$$

where

$$d_{code}^{(N,k)}(i, t) = \begin{cases} \left| \xi^{N,k}(i+t) - \xi^{N,k}(i) \right| & 1 \leq i \leq Nh_k - t \\ 1 & i > Nh_k - t \end{cases} \quad (1.5)$$

Lemma 1.7. $\lim_{N \rightarrow \infty} \delta_k^{(N)}(T^t)$ exists.

Proof. Suppose $N_0 \gg th_k$, let $\eta_k^{(t)} = \frac{1}{th_k} \sum_{i=1}^{th_k} d_{code}^{N,k}(i, t)$

For any $n > N_0$, suppose $n = tp_0 + n'$ ($0 \leq n' < t$), then

$$\begin{aligned} \delta_k^{(n)}(T^t) &= \frac{1}{nh_k} \sum_{i=1}^{nh_k} d_{code}^{(n,k)}(i, t) \\ &= \frac{tp_0}{n} \frac{1}{tp_0 h_k} (p_0 \sum_{i=1}^{th_k} d_{code}^{(n,k)}(i, t) + \sum_{i=tp_0 h_k + 1}^{nh_k} d_{code}^{(n,k)}(i, t)) \end{aligned}$$

Thus

$$\frac{tp_0}{n} \eta_k^{(t)} \leq \delta_k^{(n)}(T^t) \leq \frac{tp_0}{n} (\eta_k^{(t)} + \frac{b}{p_0 h_k})$$

Therefore, it is obvious that:

$$\lim_{n \rightarrow \infty} \delta_k^n(T^t) = \eta_k^t \tag{1.6}$$

Done □

Now let $\delta_k(T^t) = \lim_{n \rightarrow \infty} \delta_k^{(n)}(T^t) = \eta_k^{(t)}$

Lemma 1.8. $|\delta_k - \delta_{k+1}(T^t)| \leq \frac{2t}{h_{k+1}}$

Therefore $\delta_k(T^t)$ converges. Define

$$\delta(T^t) = \lim_{k \rightarrow \infty} \delta_k(T^t)$$

This derives the following proposition:

Proposition 1.9. $\delta(T^{h_k}, Id) < \frac{1}{2^k}$

Corollary 1.10. $(\mathbb{X}, \mathcal{B}, \mu, T)$ is rigid.

2 Relative Prime Relation

In this section, we set up the number theoretical relation of any positive integer with the height of the k -stack h_k . The result is more general than what is needed in section 3.

The sequence of integer h_k is also described by induction:

$$\begin{aligned} h_0 &= 1; \\ h_{k+1} &= b^{k+1} h_k + 1 \end{aligned} \tag{2.1}$$

where $b \in \mathbb{N}, b \geq 2$.

The notation $n(m), n, m \in \mathbb{N}$ is used for the integer k such that $k \equiv n \pmod{m}$, and $0 \leq k < m$.

Let $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}_q^* = \{c + q\mathbb{Z} \mid (c, q) = 1\}$, ((m, n) is the notation for the largest common divisor of m and n). Thus \mathbb{Z}_q is a finite commutative ring, and $\mathbb{Z}_q^* \subset \mathbb{Z}_q$ is the multiplicative group of multiply invertible elements of \mathbb{Z}_q .

Given $b \in \mathbb{Z}_q^*$, $b \geq 2$, we define a map from $\mathbb{Z}_q^* \times \mathbb{Z}_q$ to itself ($T : \mathbb{Z}_q^* \times \mathbb{Z}_q \rightarrow \mathbb{Z}_q^* \times \mathbb{Z}_q$), which is also understood as a skew product of the rotation on \mathbb{Z}_q^* . That is:

$$T(x, y) = T_{q,b}(x, y) = (bx, xy + 1) \quad (2.2)$$

It is well understood that $(bx, xy + 1) \in \mathbb{Z}_q^* \times R$

Lemma 2.1. *Suppose $b, q \in \mathbb{N}$, $(b, q) = 1$. then there exists $n = n(q, b) \in \mathbb{N}$ such that*

$$T^{sn} = Id, \text{ for some } s \in \mathbb{N}$$

Proof. Without loss of generality, suppose $b \in \mathbb{Z}_q^*$, $b \geq 2$. Since $\mathbb{Z}_q^* \times \mathbb{Z}_q$ is a finite set, T is just a permutation with finite order. \square

Lemma 2.2. *For any $k \in \mathbb{N}$, we have*

$$T^k(1, 0) = (b^k(q), h_{k-1}(q)) \equiv (b^k, h_{k-1}) \pmod{q}, \quad k \in \mathbb{N} \quad (2.3)$$

Proof. It is easy to see that

$$\begin{aligned} T(1, 0) &= (b(q), 1(q)) = (b(q), h_0(q)) \\ T^k(1, 0) &= T(b^{k-1}(q), h_{k-2}(q)) = (b^k(q), (b^{k-1}h_{k-2} + 1)(q)) \\ &= (b^k(q), h_{k-1}(q)) \end{aligned}$$

Therefore the lemma is proved by induction. \square

Lemma 2.3. *Suppose $(b, q) = 1$ then $h_k \equiv 0 \pmod{q}$ for infinitely many $k \in \mathbb{N}$.*

Proof. By Lemma 2.2, $T^{sn}(1, 0) = (1, 0)$.

Therefore we have

$$\begin{aligned} T^{sn}(1, 0) &= (b^{sn}(q), h_{sn-1}(q)) \equiv (1, 0) \pmod{q} \\ h_{sn+1} &\equiv 0(q), \quad s \in \mathbb{N}, n = n(b, q) \end{aligned}$$

\square

Now we can reach the following conclusion based on of Lemma 2.3:

Proposition 2.4. *Given $b \in \mathbb{N}$, $b \geq 2$, and the sequence h_k defined by 2.1. Then for any $q \in \mathbb{N}$, there exist infinitely many $k \in \mathbb{N}$ such that*

$$h_{k+1} \equiv 1 \pmod{q}$$

Proof. Suppose b is divided by q , it is done.

So we only need to investigate the case of $1 \leq d = (b, q) < q$, $q = dq'$, $1 < q' \leq q$, $(b, q') = 1$.

By Lemma 2.3, $h_k \equiv 0 \pmod{q'}$ for infinitely many $k \in \mathbb{N}$.

On the other hand for k large enough, d divides b^{k+1} thus

$$\begin{aligned} b^{k+1}h_k &\equiv 0 \pmod{q} \\ h_{k+1} &= b^{k+1}h_k + 1 \equiv 1 \pmod{q} \end{aligned}$$

\square

3 A family of totally rank one maps

Now we revisit the generalized Del Junco-Rudolph's map $(\mathbb{X}, \mathcal{B}, \mu, T)$. In this section q is a given integer greater than 1, acting as the power index of the map.

Since the heights of the stack structure satisfies $h_{k+1} = b^{k+1}h_k + 1$, Proposition 2.4 shows that for infinitely many n , $h_n \equiv 1 \pmod{q}$ for infinitely many n . We use the notion \mathcal{N}_q to denote the set of those integers, that is $\mathcal{N}_q = \{k | k \in \mathbb{N}, h_k \equiv 1 \pmod{q}\}$.

Definition 3.1 (Totally Rank One). *We say a finite measure-preserving system $(\mathbb{X}, \beta, T, \mu)$ is totally rank one, if all positive integer powers of T are rank one.*

Lemma 3.2. *Let the sequence h_k be defined by 2.1, for any $n \in \mathcal{N}_p$, and $i \neq j$, $0 \leq i, j < h_n$, we have: $qi \neq qj \pmod{h_n}$, $i \neq j$, $0 \leq i, j < h_n$.*

Proof. Since h_n is relatively prime with q . □

Now suppose B_N^* to be the base set in the N th stack column. Then B_N^* is corresponding to the cylinder set C_N with name B_N (the N -block in the recursive formula).

We know that

$$\mu(T^{h_N}(B_N^*)\Delta(B_N^*)) < \frac{1}{b^{N-1}}\mu(B_N^*) \quad (3.1)$$

and

$$\mu\left(\bigcup_{i=0}^{h_N-1} (T^i B_N^*)\right) > 1 - b h_N \mu(\mathcal{S}_k) \quad (3.2)$$

By Proposition 1.6 we have

$$\mu\left(\bigcup_{i=0}^{h_N-1} (T^i B_N^*)\right) > 1 - b^2 h_N / h_{N+1} > 1 - 1/b^{N-1} \quad (3.3)$$

Equation 3.3 shows that the stack of disjoint union of $\{T^i B_N^*\}$ is almost the whole space, except for a part of measure no more than $\frac{1}{b^{N-1}}$. Equation 3.1 tells us the first return time of B_N^* is h_N except for a set of measure no more than $\frac{1}{b^{N-1}}$. Now suppose $N \in \mathcal{N}_p$. let $A_N^* = B_N^* \cap T^{-h_N}(B_N^*) \cap T^{-2h_N}(B_N^*) \cap \dots \cap T^{-(q-1)h_N}(B_N^*)$. It is easy to see that

$$\begin{aligned} \mu(A_N^*) &\geq \mu(B_N^*) - \sum_{i=1}^{q-1} \mu(B_N^* \Delta T^{-ih_N}(B_N^*)) \\ &> \mu(B_N^*) - \frac{(q-1)(q-2)}{2} \mu(T^{h_N}(B_N^*)\Delta(B_N^*)) \end{aligned}$$

By 3.1 we know that

$$\mu(A_N^*) > \left(1 - \frac{(q-1)(q-2)}{2} \cdot \frac{1}{b^{N-1}}\right) \mu(B_N^*) \quad (3.4)$$

Define the function $\tau : \{1, 2, \dots, h_N\}$ by $\tau(i) = iq(h_N)$. Since $N \in \mathcal{N}_p$, h_N is relatively prime to q , therefore by Lemma 3.2, $\tau(i)$ is a h_N -permutation. it is obvious to see that $T^{qi}(A_N^*) \subset T^{\tau(i)}(B_N^*)$ ($0 < i \leq h_N$), therefore, we have the following 3 claims:

- (I) $\{T^{qi}(A_N^*), 0 \leq i < h_N\}$ is a collection of pairwise disjoint sets;

(II)

$$\begin{aligned}
\mu((T^q)^{h_N}(A_N^*)\Delta A_N^*) &< 2\mu(A_N^*\Delta B_N^*) + \mu(B_N^*\Delta(T^q)^{h_N}(B_N^*)) \\
&\leq (q-1)(q-2)\frac{1}{b^{N-1}}\mu(B_N^*) + \frac{q}{b^{N-1}}\mu(B_N^*) \\
&= \frac{(q^2 - 2q + 2)}{b^{N-1}}\mu(B_N^*)
\end{aligned} \tag{3.5}$$

(III) we see that (I) and (II), together with 3.3, imply

$$\begin{aligned}
\mu\left(\bigcup_{i=0}^{h_N-1}(T^q)^i(A_N^*)\right) &> \left(1 - \frac{(q-1)(q-2)}{2}\right)\frac{1}{b^{N-1}}\mu\left(\bigcup_{i=0}^{h_N-1}T^i(B_N^*)\right) \\
&= \left(1 - \frac{(q-1)(q-2)}{2}\right)\frac{1}{b^{N-1}}h_N\mu(B_N^*)
\end{aligned} \tag{3.6}$$

We know that $C_N = \bigcup_{i=0}^{h_N-1}T^i(B_N^*)$ is the union of all the levels in the N th stack, so $\mathbb{X} - C_N$ is the remainder of the spacer set taken away the set of spacers used in the first N steps during the cutting and stacking process. Thus $\mathbb{X} - C_N = \mathcal{S}_N$, $\mu(C_N) = 1 - \mu(\mathcal{S}_N) > 1 - \frac{2}{h_{N+1}}$

$$\mu\left(\bigcup_{i=0}^{h_N-1}(T^q)^i(A_N^*)\right) > \left(1 - \frac{(q-1)(q-2)}{2}\right)\frac{1}{b^{N-1}}\left(1 - \frac{2}{h_{N+1}}\right) \tag{3.7}$$

Remark. Equation 3.6 shows that the measure of $\left(\bigcup_{i=0}^{h_N-1}(T^q)^i(A_N^*)\right)$, the T^q -stack with base A_N^* , is close to the full measure, since B_N^* is the base of the N th T -stack and h_N is the corresponding height.

Now, (I), (II) and (III) tell us the following:

Theorem 3.3. *All the notations as above, every generalized Del Junco-Rudolph's map T is totally rank one.*

Remark. Though the rigidity of T^q is derived from b), it is a simple implication of the fact that all powers of a rigid automorphism on a standard Borel space are rigid.

Corollary 3.4. *In the weak closure of each general del Junco-Rudolph's map, there is a dense G_δ subset of rank one transformations.*

References

- [1] J.R. Baxter, *A class of ergodic transformations having simple spectrum*, Prodeeding of AMS, VOL 27, No. 2, 1971.
- [2] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. 22 1969 559–562.
- [3] S. Ferenczi, *Systems of finite rank*, Collaq. Math. 73 (1997), P. 35-65.
- [4] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.
- [5] G.R. Goodson, *Ergodic Dynamical Systems Conjugate to Their Composition Squares* Acta Math. Univ. Comenianae, Vol. LXXI, 2(2002), P. 201-210.

- [6] A. del Junco, D. Rudolph *A rank-one, rigid, simple, prime map*, Ergodic Theory and Dynamical Systems, (1987), 7, P.229-247.
- [7] A. del Junco, D. Rudolph *On Ergodic action whose self-joinings are graphs*, Ergodic Theory and Dynamical Systems, (1987), 7, P.531-557.
- [8] A. del Junco, *Disjointness of measure-preserving transformations*, self-joining and category. Ergodic Theory and Dynamical Systems I, Proceedings of Special Year, (1981) P. 81-89.
- [9] A.A. Prikhod'ko, V.V. Ryzhikov, Several questions and hypotheses concerning the limit polynomials for chacon transformation, arXiv:1207.0614, 2012.
- [10] W.A. Veech, *The Metric Theory of Interval Exchange Transformation I : Generic Spectral Properties*, American Journal of Mathematics, 107(6):1331-1359, 1984.
- [11] M. Viana, *Ergodic Theory of Interval Exchange maps*, Rev. Mat. Complut., 19(2006), no. 1, 7-100.
- [12] Y. Wu, *Applications of Rauzy Induction on the generic ergodic theory of interval exchange transformations*, Doctor of Philosophy Thesis, Rice University Electronic Theses and Dissertations(2006).
- [13] Y. Wu *Whirly 3-interval exchange transformations*, Ergodic Theory and Dynamical Systems, available on CJO2015. doi:10.1017/etds.2015.63.
- [14] Y. Wu, D. Li, D. Li, Y. Wang, *Totally Rank One Interval Exchange Transformations*, Arxiv, Cornell University, arXiv:1604.02638, (2016).